

# $m$ -IRREDUCIBLE NUMERICAL SEMIGROUPS

V. BLANCO<sup>†</sup> AND J.C. ROSALES<sup>‡</sup>

vblanco@ugr.es ; jrosales@ugr.es

**ABSTRACT.** In this paper we introduce the notion of  $m$ -irreducibility that extends the standard concept of irreducibility of a numerical semigroup when the multiplicity is fixed. We analyze the structure of the set of  $m$ -irreducible numerical semigroups, we give some properties of these numerical semigroups and we present algorithms to compute the decomposition of a numerical semigroups with multiplicity  $m$  into  $m$ -irreducible numerical semigroups.

## 1. INTRODUCTION

A numerical semigroup is a subset  $S$  of  $\mathbb{N}$  (here  $\mathbb{N}$  denotes the set of nonnegative integers) closed under addition, containing zero and such that  $\mathbb{N} \setminus S$  is finite.

A numerical semigroup is *irreducible* if it cannot be expressed as an intersection of two numerical semigroups containing it properly. This notion was introduced in [8] where it is also shown that the family of irreducible numerical semigroups is the union of two families of numerical semigroups with special importance in this theory: symmetric and pseudo-symmetric numerical semigroups.

Every numerical semigroup can be expressed as a finite intersection of irreducible numerical semigroups. In [7], the authors give an algorithm that allows to compute, for a numerical semigroup  $S$ , a finite set of irreducible numerical semigroups,  $S_1, \dots, S_n$ , such that  $S = S_1 \cap \dots \cap S_n$ . Furthermore, the above decomposition, given by that algorithm, is minimal in the sense that  $n$  is the smallest possible positive integer with that property. In [5], upper and lower bounds are given for that  $n$  in terms of  $S$ .

If  $S$  is a numerical semigroup, the least positive integer belonging to  $S$  is called the *multiplicity* of  $S$  and we denote it by  $m(S)$ .

Let  $m$  be a positive integer and  $\mathcal{S}(m)$  the set of numerical semigroups with multiplicity  $m$ . It is clear that  $(\mathcal{S}(m), \cap)$  is a semilattice, that is, it is a semigroup where all its elements are idempotent. The search of the generators of this semilattice leads us to the following definition. An element  $S$  in  $\mathcal{S}(m)$  is  $m$ -irreducible if it cannot be expressed as the intersection of two elements in  $\mathcal{S}(m)$  containing it properly. In Section 2 we prove that every element in  $\mathcal{S}(m)$  can be expressed as a finite intersection of  $m$ -irreducible numerical semigroups. Furthermore, we prove that  $\{S : S \text{ is a } m\text{-irreducible numerical semigroup}\}$  coincides with  $\{\{x \in \mathbb{N} : x \geq m\} \cup \{0\}\} \cup \{\{x \in \mathbb{N} : x \geq m \text{ and } x \neq i\} \cup \{0\} : i = m + 1, \dots, 2m - 1\} \cup \{S : S \text{ is an irreducible numerical semigroup and } m(S) = m\}$ .

If  $S$  is a numerical semigroup, the largest integer not belonging to  $S$  is called the *Frobenius number* of  $S$  and we denote it by  $F(S)$ . We say that a positive integer,  $x$ , is a *gap* of  $S$  if  $x \notin S$ . We denote by  $G(S)$  the set of all the gaps of  $S$ . The cardinal of  $G(S)$  is called the *genus* of  $S$ , and we denote it by  $g(S)$ . We can find in the literature several characterization for irreducible numerical semigroups. In [9], the authors state that the irreducible numerical semigroups are those with minimum genus among all the numerical semigroups with the same Frobenius number. In Section 3 we prove that this characterization is also valid for  $m$ -irreducible numerical semigroups. In fact, we prove that an element in  $\mathcal{S}(m)$  is  $m$ -irreducible if and only if it has minimum genus in the set of all the elements in  $\mathcal{S}(m)$  with the same Frobenius number.

*Date:* June 18, 2010.

*2010 Mathematics Subject Classification.* 20M14, 13H10.

*Key words and phrases.* numerical semigroup, irreducible numerical semigroup, multiplicity, Frobenius number.

<sup>†</sup>The first author has been supported by the Juan de la Cierva program (Ref. JCI-2009-03896) and project MTM2007-67433-C02-01

<sup>‡</sup>The second author has been supported by the project MTM2007-62346.

Given a numerical semigroup  $S$  with multiplicity  $m$ , we denote by  $\mathcal{O}_m(S) = \{S' \in \mathcal{S}(m) : S \subseteq S'\}$  the set of *oversemigroups* with multiplicity  $m$  of  $S$ . The aim of Section 4 is to provide an algorithm to compute  $\mathcal{O}_m(S)$  when  $S$  is given. These results are used in Section 5 to give an algorithm to compute a decomposition, as an intersection of  $m$ -irreducible numerical semigroups, of a numerical semigroup with multiplicity  $m$ . Moreover, the decomposition given by the algorithm is minimal in the sense that it uses the smallest number of  $m$ -irreducible numerical semigroups for taking part of the decomposition.

For concluding this introduction, observe that although this work is performed by a “semigroupist” point of view, it can be used in Commutative Ring Theory. In fact, let  $M$  be a submonoid of  $(\mathbb{N}, +)$ ,  $\mathbb{K}$  a field, and  $\mathbb{K}[[t]]$  the ring of formal power series over  $\mathbb{K}$ . It is well-known (see for instance, [2]) that  $\mathbb{K}[[M]] = \left\{ \sum_{s \in M} a_s t^s : a_s \in \mathbb{K} \right\}$  is a subring of  $\mathbb{K}[[t]]$ , called the ring of the semigroup associated to  $M$ .

All the above invariants, as the multiplicity, the genus (degree of singularity) and the Frobenius number (conductor minus one) have their corresponding interpretation in this context (see [2]). Moreover, in [4] it is shown that a numerical semigroup is symmetric if and only if  $\mathbb{K}[[S]]$  is a Gorenstein ring and in [3] the rings  $\mathbb{K}[[S]]$  when  $S$  is pseudo-symmetric are called Kunz rings. Then, as a consequence of the results in this paper we have that, given a numerical semigroup  $S$  we can decompose the ring  $\mathbb{K}[[S]]$  as intersection of rings with the same multiplicity where some of them are Gorenstein, some other are Kunz and others are rings associated to numerical semigroups with special simplicity:  $\{x \in \mathbb{N} : x \geq m\} \cup \{0\}$  and  $\{x \in \mathbb{N} : x \geq m \text{ and } x \neq i\} \cup \{0\}$  for  $i \in \{m+1, \dots, 2m-1\}$ .

## 2. $m$ -IRREDUCIBLE NUMERICAL SEMIGROUPS

We begin this section showing that every numerical semigroup with multiplicity  $m$  can be expressed as a finite intersection of  $m$ -irreducible numerical semigroups.

Note that  $\{x \in \mathbb{N} : x \geq m\} \cup \{0\}$  is the maximum (with respect to the inclusion ordering) of  $\mathcal{S}(m)$  and then, this semigroup is  $m$ -irreducible.

**Proposition 1.** *Let  $S \in \mathcal{S}(m)$ . Then, there exist  $S_1, \dots, S_k$   $m$ -irreducible numerical semigroups such that  $S = S_1 \cap \dots \cap S_k$ .*

*Proof.* We prove the result by induction over  $g(S)$ . If  $g(S) = m-1$ , then  $S = \{x \in \mathbb{N} : x \geq m\} \cup \{0\}$  and then  $m$ -irreducible. Assume that  $g(S) \geq m$  and that  $S$  is no  $m$ -irreducible. Then there exist  $S_1$  and  $S_2$  in  $\mathcal{S}(m)$  such that  $S \subsetneq S_1$ ,  $S \subsetneq S_2$ , and  $S = S_1 \cap S_2$ . By the induction hypothesis, there exist  $S_{11}, \dots, S_{1p}, S_{21}, \dots, S_{2q}$   $m$ -irreducible numerical semigroups such that  $S_1 = S_{11} \cap \dots \cap S_{1p}$  and  $S_2 = S_{21} \cap \dots \cap S_{2q}$ . Then,  $S = S_{11} \cap \dots \cap S_{1p} \cap S_{21} \cap \dots \cap S_{2q}$  is a decomposition of  $S$  into  $m$ -irreducible numerical semigroups.  $\square$

Our next goal for this section is to prove Theorem 3 which states that the  $m$ -irreducible numerical semigroups are those maximal elements in  $\mathcal{S}(m)$  with the additional condition that they have certain Frobenius number. In order to prove that result, we first introduce some notation and previous results.

Given two positive integer  $m$  and  $F$ , we denote by  $\mathcal{S}(m, F)$  the set of numerical semigroups with multiplicity  $m$  and Frobenius number  $F$ . Note that  $\mathcal{S}(m, F) \neq \emptyset$  if and only if  $F \geq m-1$  and  $F$  is not a multiple of  $m$ .

Denote by  $\mathcal{S}^*(m, F)$  the set of maximal elements in  $\mathcal{S}(m, F)$  (with respect to the inclusion ordering).

The following result has an immediate proof and it is left to the reader

**Lemma 2.**

- (1) *If  $S \neq \mathbb{N}$  is a numerical semigroup, then  $S \cup \{F(S)\}$  is also a numerical semigroup.*
- (2) *Let  $S_1, \dots, S_n$  be numerical semigroups and  $S = S_1 \cap \dots \cap S_n$ . Then,  $F(S) = \max\{F(S_1), \dots, F(S_n)\}$ .*

We are now ready to prove the announced result.

**Theorem 3.** *Let  $S \in \mathcal{S}(m)$ . Then,  $S$  is  $m$ -irreducible if and only if  $S \in \mathcal{S}^*(m, F(S))$ .*

*Proof.* (Necessity) If  $S = \{x \in \mathbb{N} : x \geq m\} \cup \{0\}$ , clearly  $S \in \mathcal{S}^*(m, F(S))$ . Assume that  $S \neq \{x \in \mathbb{N} : x \geq m\} \cup \{0\}$ , then  $F(S) > m$ . By Lemma 2 we have that  $S \cup \{F(S)\} \in \mathcal{S}(m)$ . If  $S \notin \mathcal{S}^*(m, F(S))$  there exists  $T \in \mathcal{S}(m, F(S))$  such that  $S \subsetneq T$ . It is clear that  $S = T \cap (S \cup \{F(S)\})$  contradicting that  $S$  is  $m$ -irreducible.

(Sufficiency) Let  $S \in \mathcal{S}^*(m, F(S))$ . If  $S$  is not  $m$ -irreducible then, there exist  $S_1$  and  $S_2$  in  $\mathcal{S}(m)$  such that  $S \subsetneq S_1$ ,  $S \subsetneq S_2$  and  $S = S_1 \cap S_2$ . By Lemma 2,  $F(S) \in \{F(S_1), F(S_2)\}$ . Assume without loss of generality that  $F(S) = F(S_1)$ . Then,  $S_1 \in \mathcal{S}(m, F(S))$  and  $S \subsetneq S_1$  contradicting the maximality of  $S$ .  $\square$

In what follows we intend to give a proof for Proposition 6 where we describe explicitly the elements in  $\mathcal{S}^*(m, F)$ . Before that, we give some previous results. The following lemma is well-known and appears in [9].

**Lemma 4.** *Let  $S$  be a numerical semigroup and assume that  $h = \max\{x \in \mathbb{Z} \setminus S : F(S) - x \notin S, x \neq \frac{F}{2}\}$  exists, then  $S \cup \{h\}$  is a numerical semigroup with Frobenius number  $F(S)$ .*

In [8] it is shown that a numerical semigroup  $S$  is irreducible if and only if it is maximal in the set of numerical semigroups with Frobenius number  $F(S)$ . As a direct consequence of the above lemma we get the following result.

**Lemma 5.** *A numerical semigroup is irreducible if and only if  $\{x \in \mathbb{Z} \setminus S : F(S) - x \notin S \text{ and } x \neq \frac{F(S)}{2}\} = \emptyset$ .*

We are ready to describe the structure of  $\mathcal{S}^*(m, F)$ . Here, we denote by  $a \equiv b \pmod{c}$  if  $a - b$  is multiple of  $c$ .

**Proposition 6.** *Let  $m$  and  $F$  be positive integers such that  $F \geq m - 1$  and  $F \not\equiv 0 \pmod{m}$ .*

- (1) *If  $F = m - 1$ , then  $\mathcal{S}^*(m, F) = \{\{x \in \mathbb{N} : x \geq m\} \cup \{0\}\}$ .*
- (2) *If  $m < F < 2m$ , then  $\mathcal{S}^*(m, F) = \{\{x \in \mathbb{N} : x \geq m, x \neq F\} \cup \{0\}\}$ .*
- (3) *If  $F > 2m$ , then  $\mathcal{S}^*(m, F) = \{S \in \mathcal{S}(m, F) : S \text{ is irreducible}\}$ .*

*Proof.* (1) and (2) are trivial. Let us then prove (3). It is clear, by the comment before Lemma 5, that  $\{S \in \mathcal{S}(F, m) : S \text{ is irreducible}\} \subseteq \mathcal{S}^*(m, F)$ . We prove the other inclusion. Let  $S \in \mathcal{S}^*(m, F)$ . If  $S$  is not irreducible, we know, by Lemma 5, that  $h = \max\{x \notin S : F(S) - x \notin S, x \neq \frac{F}{2}\}$  exists, and by Lemma 4, that  $S \cup \{h\}$  is numerical semigroup with Frobenius number  $F$ . Furthermore,  $h > \frac{F}{2} > m$  and then,  $S \cup \{h\} \in \mathcal{S}(m, F)$  contradicting the maximality of  $S$ .  $\square$

As a consequence of Theorem 3 and Proposition 6 we get this corollary.

**Corollary 7.** *A numerical semigroup,  $S$ , with multiplicity  $m$  is  $m$ -irreducible if and only if one of the following conditions holds:*

- (1)  $S = \{x \in \mathbb{N} : x \geq m\} \cup \{0\}$ .
- (2)  $S = \{x \in \mathbb{N} : x \geq m, x \neq f\} \cup \{0\}$  with  $f \in \{m + 1, \dots, 2m - 1\}$ .
- (3)  $S$  is an irreducible numerical semigroup.

### 3. THE GAPS OF A $m$ -IRREDUCIBLE NUMERICAL SEMIGROUP

Let  $q$  be a rational number. We denote by  $\lceil q \rceil = \min\{z \in \mathbb{Z} : q \leq z\}$  the ceiling part of  $q$ . It is well-known (see for instance [9]) that if  $S$  is a numerical semigroup then  $g(S) \geq \left\lceil \frac{F(S) + 1}{2} \right\rceil$ . The following result is easily deduced from Lemma 5 and appears in [9].

**Lemma 8.** *A numerical semigroup  $S$  is irreducible if and only if  $g(S) = \left\lceil \frac{F(S) + 1}{2} \right\rceil$ .*

As a consequence of Theorem 3, Proposition 6 and Lemma 8 we get the following result.

**Proposition 9.** *If  $S$  is a  $m$ -irreducible numerical semigroup, then*

$$g(S) = \begin{cases} m - 1 & \text{if } F(S) = m - 1, \\ m & \text{if } m < F(S) < 2m, \\ \left\lceil \frac{F(S) + 1}{2} \right\rceil & \text{if } F(S) > 2m \end{cases}.$$

By Lemma 8, the irreducible numerical semigroups are those with the smallest possible number of gaps once the Frobenius number is fixed. In the following result we prove that this property is extendable for  $m$ -irreducible numerical semigroups.

Let  $m$  and  $F$  be two positive integers such that  $F \geq m - 1$  and  $F \not\equiv 0 \pmod{m}$ . We denote by

$$g(m, F) = \min\{g(S) : S \in \mathcal{S}(m, F)\},$$

the minimum genus among all the numerical semigroups with multiplicity  $m$  and Frobenius number  $F$ .

**Theorem 10.** *Let  $S$  be a numerical semigroup with multiplicity  $m$  and Frobenius number  $F$ . Then,  $S$  is  $m$ -irreducible if and only if  $g(S) = g(m, F)$ .*

*Proof.* (Sufficiency) It is clear that if  $g(S) = g(m, F)$  then  $S$  is maximal in  $\mathcal{S}(m, F)$ . Hence,  $S \in \mathcal{S}^*(m, F)$  and by Theorem 3 we have that  $S$  is  $m$ -irreducible.

(Necessity) By Theorem 3 and Proposition 6 we distinguish the following three cases:

- (1) If  $S = \{x \in \mathbb{N} : x \geq m\} \cup \{0\}$ , it is clear that  $g(S) = m - 1 = g(m, F)$ .
- (2) If  $S = \{x \in \mathbb{N} : x \geq m, x \neq F\} \cup \{0\}$  with  $m < F < 2m$ , then, clearly  $g(S) = m = g(m, F)$ .
- (3) If  $S$  is irreducible and  $F > 2m$ , then by Lemma 8,  $g(S) = g(m, F)$ .

□

From the above theorem we have that the  $m$ -irreducible numerical semigroups are exactly those with minimum genus among all the numerical semigroups with its same multiplicity and Frobenius number.

**Corollary 11.** *Let  $S$  be a numerical semigroup with multiplicity  $m$ . Then,  $S$  is  $m$ -irreducible if and only if  $g(S) \in \{m - 1, m, \left\lceil \frac{F(S) + 1}{2} \right\rceil\}$ .*

*Proof.* (Necessity) It is a direct consequence of Proposition 9.

(Sufficiency) If  $g(S) = m - 1$ , then  $S = \{x \in \mathbb{N} : x \geq m\} \cup \{0\}$  and by Corollary 7 we have that  $S$  is  $m$ -irreducible.

If  $g(S) = m$  then we deduce that  $m < F(S) < 2m$ , and  $S = \{x \in \mathbb{N} : x \geq m, x \neq F(S)\} \cup \{0\}$  and again by applying Corollary 7,  $S$  is  $m$ -irreducible.

If  $g(S) = \left\lceil \frac{F(S) + 1}{2} \right\rceil$ , then by Corollary 7 and Lemma 8, we conclude that  $S$  is  $m$ -irreducible. □

Let  $S$  be a numerical semigroup. We say that an element  $x \in \mathbb{N} \setminus S$  is a special gap of  $S$  if  $S \cup \{x\}$  is a numerical semigroup. We denote by  $\text{SG}(S)$  the set of all the special gaps of  $S$ . Let  $A$  be a set, we denote by  $|A|$  the cardinal of  $A$ .

The following result appears in [9].

**Lemma 12.** *Let  $S$  be a numerical semigroup.*

- (1)  $S$  is irreducible if and only if  $|\text{SG}(S)| \leq 1$ .
- (2) If  $T$  is a numerical semigroup such that  $S \subsetneq T$  and  $x = \max(T \setminus S)$ , then  $x \in \text{SG}(S)$ .

From the above lemma we have that the irreducible numerical semigroups are characterized by the cardinal of the set of its special gaps. Next, we show that the  $m$ -irreducible numerical semigroups are characterized by the cardinal of its special gaps that are greater than  $m$ .

**Theorem 13.** *Let  $S$  be a numerical semigroup with multiplicity  $m$ . Then,  $S$  is  $m$ -irreducible if and only if  $|\{x \in \text{SG}(S) : x > m\}| \leq 1$ .*

*Proof.* (Necessity) If  $S$  is a  $m$ -irreducible numerical semigroup, then, by Theorem 3 and Proposition 6, one of the following condition holds:

- (1) If  $S = \{x \in \mathbb{N} : x \geq m\} \cup \{0\}$ , it is clear that  $|\{x \in \text{SG}(S) : x > m\}| = 0 \leq 1$ .
- (2) If  $S = \{x \in \mathbb{N} : x \geq m, x \neq F(S)\} \cup \{0\}$  with  $m < F(S) < 2m$ , then, clearly  $|\{x \in \text{SG}(S) : x > m\}| = |\{F(S)\}| = 1$ .
- (3) If  $S$  is irreducible and  $F(S) > 2m$ , then by Lemma 12,  $|\{x \in \text{SG}(S) : x > m\}| \leq 1$ .

(Sufficiency) If  $S$  is not an  $m$ -irreducible numerical semigroup then there exist  $S_1, S_2 \in \mathcal{S}(m)$  such that  $S \subsetneq S_1$ ,  $S \subsetneq S_2$  and  $S = S_1 \cap S_2$ . For  $i \in \{1, 2\}$ , let  $x_i = \max(S_i \setminus S)$ . Because  $S = S_1 \cap S_2$  we have that  $x_1 \neq x_2$  and by Lemma 12, we conclude that  $|\{x \in \text{SG}(S) : x > m\}| \geq 2$  contradicting the hypothesis. □

In the rest of the section we denote by  $T$  a numerical semigroup and  $n \in T \setminus \{0\}$ . The Apéry set [1] of  $T$  with respect to  $n$  is  $\text{Ap}(T, n) = \{t \in T : t - n \notin T\}$ . Our goal is to show how to determine  $\text{SG}(T)$  when  $\text{Ap}(T, n)$  is known. This process will be useful for the rest of this paper.

It is well-known and easy to prove that  $\text{Ap}(T, n) = \{w(0), \dots, w(n-1)\}$ , where  $w(i)$  is the least element in  $T$  congruent with  $i$  modulo  $n$ . If we denote by  $a \bmod b$  the remainder of division of  $a$  by  $b$ , then a positive integer,  $x$ , is in  $T$  if and only if  $w(x \bmod n) \leq x$  (see for instance [9]).

By using the notation introduced in [6], an integer  $x$  is a pseudo-Frobenius number of  $T$  if  $x \notin T$  and  $x + t \in T$  for all  $t \in T \setminus \{0\}$ . We denote by  $\text{PF}(T)$  the set of all pseudo-Frobenius number of  $T$ . The cardinality of the above set is an important invariant of  $T$ , called the *type* of  $T$  (see [2]). The relationship between  $\text{PF}(T)$  and  $\text{SG}(T)$  is shown in the following lemma, whose proof is immediate.

**Lemma 14.** *With the above conditions  $\text{SG}(T) = \{x \in \text{PF}(T) : 2x \in T\}$ .*

Over  $\mathbb{N}$ , we can define the following order relation:

$$a \leq_T b \text{ if } b - a \in T$$

The following result appears in [6] and characterizes the set of pseudo-Frobenius numbers of a numerical semigroup in terms of one of its Apéry sets.

**Lemma 15.** *With the above notation  $\text{PF}(T) = \{w - n : w \in \text{Maximals}_{\leq_T} \text{Ap}(T, n)\}$ .*

Note that  $w(i) \in \text{Maximals}_{\leq_T} \text{Ap}(T, n)$  if and only if  $w(k) - w(i) \notin \text{Ap}(T, n)$  for all  $k \neq i$ .

We finish this section by giving an algorithm that allows to determine the special gaps of a numerical semigroup when the Apéry set with respect a non-zero element of the semigroup is given. Algorithm 1 shows a pseudocode for this computation.

---

**Algorithm 1:** Computing the special gaps when an Apéry set is given.

---

**Input** : A numerical semigroup  $T$  and its Apéry set with respect to  $n \in T \setminus \{0\}$ ,  
 $\text{Ap}(T, n) = \{w(0), \dots, w(n-1)\}$ .  
 Compute  $M = \{w(i) - n : w(k) - w(i) \notin \text{Ap}(T, n), \text{ for all } k \neq i\}$ .  
**Output:**  $\text{SG}(T) = \{m \in M : 2m \geq w(2m \bmod n)\}$ .

---

Next, we illustrate the usage of the above algorithm in a simple example.

**Example 16.** *Let us compute the special gaps of the numerical semigroup  $S = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$  by using Algorithm 1. It is clear that  $\text{Ap}(S, 5) = \{0, 7, 9, 16, 18\}$ . Then,  $M = \{16 - 5 = 11, 18 - 5 = 13\}$  and  $\text{SG}(S) = \{11, 13\}$ .*

#### 4. THE OVERSEMIGROUPS OF MULTIPLICITY $m$ OF A NUMERICAL SEMIGROUP

Let  $S$  be a numerical semigroups with multiplicity  $m$ . We denote by  $\mathcal{O}_m(S) = \{S' \in \mathcal{S}(m) : S \subseteq S'\}$  the set of oversemigroups with multiplicity  $m$  of  $S$ .

The elements in  $\mathcal{O}_m(S)$  can be obtained recursively. The main idea for this construction is the following result that can be directly obtained from Lemma 12.

**Lemma 17.** *Let  $S$  and  $S'$  be elements in  $\mathcal{S}(m)$  such that  $S \subsetneq S'$  and let  $x = \max(S' \setminus S)$ . Then  $S \cup \{x\} \in \mathcal{S}(m)$ .*

Given two numerical semigroups  $S$  and  $S'$  in  $\mathcal{S}(m)$  with  $S \subsetneq S'$ , we define recursively the following sequence of elements in  $\mathcal{S}(m)$ :

- $S_0 = S$ .
- $S_{n+1} = \begin{cases} S_n & \text{if } S_n = S', \\ S_n \cup \{\max(S' \setminus S_n)\} & \text{otherwise.} \end{cases}$

It is clear that if  $|S' \setminus S| = k$ , then  $S = S_0 \subsetneq S_1 \subsetneq \dots \subsetneq S_k = S'$ .

With this idea it is not difficult to design an algorithm that allows to compute all the elements in  $\mathcal{O}_m(S)$  for a given numerical semigroup with multiplicity  $m$ ,  $S$ . The main idea for this procedure is that if

$S' \in \mathcal{O}_m(S)$  (we start with  $S' = S$ ) and  $\{x_1, \dots, x_r\} = \{x \in \text{SG}(S') : x > m\}$  then  $S' \cup \{x_1\}, \dots, S' \cup \{x_r\}$  are also in  $\mathcal{O}_m(S)$ . Algorithm 2 shows a pseudocode for this procedure.

---

**Algorithm 2:** Computing the oversemigroups with multiplicity  $m$  of a numerical semigroup.

---

**Input** : A numerical semigroup  $S$  with multiplicity  $m$   
**Initialization:**  $A = \{S\}$  and  $B = \{S\}$ .  
**while**  $B \neq \emptyset$  **do**  
    • For each  $S' \in B$  compute  $D(S') = \{x \in \text{SG}(S') : x > m\}$ .  
    • Set  $A := A \cup \{\bigcup_{S' \in B} \{S' \cup \{x\} : x \in D(S')\}\}$ .  
    • Set  $B := \bigcup_{S' \in B} \{S' \cup \{x\} : x \in D(S')\}$ .  
**Output** :  $A = \mathcal{O}_m(S)$ .

---

Observe that the difficulty of the above algorithm lies on the computation of  $\{x \in \text{SG}(S') : x > m\}$ . Recall that if we know the Apéry set  $\text{Ap}(S, m)$ , by using Algorithm 1 we can easily compute the set  $\{x \in \text{SG}(S') : x > m\}$ .

The following result states that if we know  $\text{Ap}(S, m)$ , then we can also compute  $\text{Ap}(S', m)$  for all those oversemigroups  $S'$  that appears when we run Algorithm 2.

**Lemma 18.** *Let  $T \in \mathcal{S}(m)$  and  $\text{Ap}(T, m) = \{w(0), \dots, w(m-1)\}$ . If  $x \in \text{SG}(T)$ , then*

$$\text{Ap}(T \cup \{x\}, m) = (\text{Ap}(T, m) \setminus \{w(x \bmod m)\}) \cup \{w(x \bmod m) - m\}$$

*Proof.* Note that by Lemma 15 we know that  $x = w(x \bmod m) - m$ . □

Observe that a numerical semigroup with multiplicity  $m$ ,  $S$ , is completely determined by  $\text{Ap}(S, m)$ . Hence,  $S$  is also completely determined by a  $(m-1)$ -tuple  $(w(1), \dots, w(m-1))$  in  $\mathbb{N}^{m-1}$  where  $w(i)$  is the least element in  $S$  congruent with  $i$  modulo  $m$ . We will call in the rest of the paper to  $(w(1), \dots, w(m-1))$  the *coordinates* of  $S$ .

We denote by  $[m-1] = \{1, \dots, m-1\}$ , and for each  $i \in [m-1]$  by  $e_i$ , the  $(m-1)$ -tuple having a 1 as its  $i$ th entry and zeros otherwise. With this notation, the following result is a reformulation of Lemma 18.

**Lemma 19.** *Let  $T \in \mathcal{S}(m)$  and  $x \in \text{SG}(T)$  such that  $x > m$ . If  $(x_1, \dots, x_{m-1})$  are the coordinates of  $T$ , then  $(x_1, \dots, x_{m-1}) - m \cdot e_{x \bmod m}$  are the coordinates of  $T \cup \{x\}$ .*

The reader can easily check that the following algorithm computes, from the coordinates of a numerical semigroup with multiplicity  $m$ ,  $S$ , the set of all the coordinates of the elements in  $\mathcal{O}_m(S)$ .

---

**Algorithm 3:** The coordinates of all the oversemigroups with multiplicity  $m$  of a numerical semigroup.

---

**Input** : The coordinates  $x = (x_1, \dots, x_{m-1})$  of a numerical semigroup  $S$  with multiplicity  $m$   
**Initialization:**  $A = \{x\}$  and  $B = \{x\}$ .  
**while**  $B \neq \emptyset$  **do**  
    • For each  $y = (y_1, \dots, y_{m-1}) \in B$  compute  
         $D(y) = \{i \in [m-1] : y_i > 2m \text{ and } y_k - y_i \notin \{y_1, \dots, y_{m-1}\} \text{ for all } k \in [m-1]\}$ .  
    • Set  $A := A \cup \{\bigcup_{y \in B} \{y - m \cdot e_i : i \in D(y)\}\}$ .  
    • Set  $B := \bigcup_{y \in B} \{y - m \cdot e_i : i \in D(y)\}$ .  
**Output** :  $A = \text{Coordinates of } \mathcal{O}_m(S)$ .

---

We finish the section by illustrating the above algorithm with an example.

**Example 20.** *Let  $S = \{0, 5, 7, 9, 10, 12, 14, \dots\}$ . It is clear that  $S$  is a numerical semigroup with multiplicity 5 and that  $(16, 7, 18, 9)$  are its coordinates.*

- We start by initializing  $A = \{(16, 7, 18, 9)\}$  and  $B = \{(16, 7, 18, 9)\}$ .  
 $D(16, 7, 18, 9) = \{1, 3\}$ .
- $A = \{(16, 7, 18, 9), (11, 7, 18, 9), (16, 7, 13, 9)\}$  and  $B = \{(11, 7, 18, 9), (16, 7, 13, 9)\}$ .  
 $D(11, 7, 18, 9) = \{3\}$  and  $D(16, 7, 13, 9) = \{1, 3\}$ .



- $A = \{(16, 7, 18, 9), (11, 7, 18, 9), (16, 7, 13, 9), (11, 7, 13, 9), (16, 7, 8, 9)\}$  and  $B = \{(11, 7, 13, 9), (16, 7, 8, 9)\}$ .  
 $D(11, 7, 13, 9) = \{1, 3\}$  and  $D(16, 7, 8, 9) = \{1\}$ .
- $A = \{(16, 7, 18, 9), (11, 7, 18, 9), (16, 7, 13, 9), (11, 7, 13, 9), (16, 7, 8, 9), (6, 7, 13, 9), (11, 7, 8, 9)\}$  and  $B = \{(6, 7, 13, 9), (11, 7, 8, 9)\}$ .  
 $D(6, 7, 13, 9) = \{3\}$  and  $D(11, 7, 8, 9) = \{1\}$ .
- $A = \{(16, 7, 18, 9), (11, 7, 18, 9), (16, 7, 13, 9), (11, 7, 13, 9), (16, 7, 8, 9), (6, 7, 13, 9), (11, 7, 8, 9), (6, 7, 8, 9)\}$  and  $B = \{(6, 7, 8, 9)\}$ .  
 $D(6, 7, 8, 9) = \emptyset$ .
- $A = \{(16, 7, 18, 9), (11, 7, 18, 9), (16, 7, 13, 9), (11, 7, 13, 9), (16, 7, 8, 9), (6, 7, 13, 9), (11, 7, 8, 9), (6, 7, 8, 9)\}$  and  $B = \emptyset$ .

And then, the coordinates of all the oversemigroups with multiplicity 5 of  $S$  are  $\{(16, 7, 18, 9), (11, 7, 18, 9), (16, 7, 13, 9), (11, 7, 13, 9), (16, 7, 8, 9), (6, 7, 13, 9), (11, 7, 8, 9), (6, 7, 8, 9)\}$

### 5. DECOMPOSITION INTO $m$ -IRREDUCIBLE NUMERICAL SEMIGROUPS

Let  $S$  be a numerical semigroup with multiplicity  $m$ . We denote by  $\mathcal{J}_m(S) = \{S' \in \mathcal{O}_m(S) : S' \text{ is } m\text{-irreducible}\}$ . From Proposition 1 we deduce that  $S = \bigcap_{S' \in \mathcal{J}_m(S)} S'$ . The following result has an immediate proof.

**Lemma 21.** *Let  $S \in \mathcal{S}(m)$  and  $\{S_1, \dots, S_n\}$  the set of all the minimal elements (with respect to the inclusion ordering) of  $\mathcal{J}_m(S)$ . Then  $S = S_1 \cap \dots \cap S_n$ .*

As a consequence of the results in Section 3 and Theorem 13 we have justified the following algorithm that computes the minimal elements in  $\mathcal{J}_m(S)$ .

---

**Algorithm 4:** Computation of  $\text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$ .

---

**Input** : A numerical semigroup  $S$  with multiplicity  $m$   
**Initialization:**  $A = \{S\}$  and  $B = \emptyset$ .  
**while**  $A \neq \emptyset$  **do**  
    For each  $S' \in A$  compute  $D(S') = \{x \in \text{SG}(S') : x > m\}$ .  
    **if**  $|D(S')| \leq 1$  **then**  
        Set  $B := B \cup \{S'\}$  and  $A := A \setminus \{S'\}$ .  
    **else**  
        Set  $A := (A \setminus \{S'\}) \cup \{S' \cup \{x\} : x \in D(S') \text{ and } S' \cup \{x\} \text{ does not contain any element of } B\}$ .  
**Output** :  $B = \text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$

---

We illustrate the above algorithm with the following example. Note that if  $S$  and  $T$  are elements in  $\mathcal{S}(m)$  with coordinates  $x$  and  $y$ , respectively, then  $S \subseteq T$  if and only if  $y \leq x$ .

**Example 22.** *Let  $S = \{0, 5, 7, 9, 10, 12, 14, \rightarrow\}$  the numerical semigroup given in Example 20. We compute the set  $\text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$ . Recall that  $(16, 7, 18, 9)$  are the coordinates of  $S$ . Then, by running Algorithm 4 we obtain:*

- $A = \{(16, 7, 18, 9)\}$  and  $B = \emptyset$ .  $D(16, 7, 18, 9) = \{11, 13\}$ .
- $A = \{(11, 7, 18, 9), (16, 7, 13, 9)\}$  and  $B = \emptyset$ .  $D(11, 7, 18, 9) = \{13\}$ .
- $A = \{(16, 7, 13, 9)\}$  and  $B = \{(11, 7, 18, 9)\}$ .  $D(16, 7, 13, 9) = \{11, 8\}$ .
- $A = \{(16, 7, 8, 9)\}$  and  $B = \{(11, 7, 18, 9)\}$ .  $D(16, 7, 8, 9) = \{11\}$ .
- $A = \emptyset$  and  $B = \{(11, 7, 18, 9), (16, 7, 8, 9)\}$ .

Note that the decomposition described in Lemma 21 is not necessarily minimal, in the sense that the smallest number of  $m$ -irreducible numerical semigroups are taking part of the decomposition. An example of this fact is shown in Example 27.

To compute the minimal decomposition are necessary the two following results.

**Lemma 23.** *Let  $S \in \mathcal{S}(m)$ . If  $S = S_1 \cap \dots \cap S_n$  with  $S_1, \dots, S_n \in \mathcal{J}_m(S)$ , then there exist  $S'_1, \dots, S'_n \in \text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$  such that  $S = S'_1 \cap \dots \cap S'_n$ .*

*Proof.* Take, for each  $i \in \{1, \dots, n\}$   $S'_i$  the element in  $\text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$  such that  $S'_i \subseteq S_i$ .  $\square$

**Lemma 24.** *Let  $S \in \mathcal{S}(m)$  and  $S_1, \dots, S_n \in \mathcal{O}_m(S)$ . Then,  $S = S_1 \cap \dots \cap S_n$  if and only if for all  $h \in \{x \in \text{SG}(S) : x > m\}$  there exists  $i \in \{1, \dots, n\}$  such that  $h \notin S_i$ .*

*Proof.* It is a direct consequence of Lemma 12.  $\square$

**Remark 25.** *Note that as a direct consequence of the above lemma we have that if  $S \in \mathcal{S}(m)$ , then  $S$  can be expressed as an intersection less or equal than  $|\{x \in \text{SG}(S) : x > m\}|$   $m$ -irreducible numerical semigroups. In fact, by Lemma 14 and Corollary 1.23 in [9] we get that we can decompose  $S$  into less or equal than  $m - 1$   $m$ -irreducible numerical semigroups.*

Assume that  $\text{Minimals}_{\subseteq}(\mathcal{J}_m(S)) = \{S_1, \dots, S_n\}$ . For each  $i \in \{1, \dots, n\}$ , let  $P(S_i) = \{h \in \text{SG}(S) : h > m \text{ and } h \notin S_i\}$ . By Lemma 24, we know that  $S = S_{i_1} \cap \dots \cap S_{i_r}$  if and only if  $P(S_{i_1}) \cup \dots \cup P(S_{i_r}) = \{x \in \text{SG}(S) : x > m\}$ . This comment and Lemma 23 justify the following algorithm that allows to compute from a given numerical semigroup with multiplicity  $m$ , a minimal decomposition as intersection of  $m$ -irreducible numerical semigroups.

---

**Algorithm 5:** Computation of a minimal decomposition of a numerical semigroup with multiplicity  $m$  in  $m$ -irreducible numerical semigroups.

---

**Input** : A numerical semigroup  $S$  with multiplicity  $m$

- (1) Compute  $\text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$  by Algorithm 4.
- (2) For each  $S' \in \text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$ , compute  $P(S') = \{h \in \text{SG}(S) : h > m \text{ and } h \notin S'\}$ .
- (3) Choose  $A \subseteq \text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$  with minimal cardinality and such that  $\bigcup_{S' \in A} P(S') = \{x \in \text{SG}(S) : x > m\}$ .

**Output:**  $A = \{S_1, \dots, S_n\}$  such that  $S = S_1 \cap \dots \cap S_n$  is a minimal decomposition of  $S$  in  $m$ -irreducible numerical semigroups.

---

The following two examples shows the usage of the above methodology.

**Example 26.** *Let  $S = \{0, 5, 10, 11, 14, 15, 16, 19, 20, 21, 22, 24, \rightarrow\}$ . The coordinates for  $S$  are  $x = (11, 22, 28, 14)$ . By using Algorithm 4, we obtain that the coordinates of the elements in  $\text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$  are*

$$(11, 22, 8, 14), (11, 22, 13, 9), (11, 17, 28, 14)$$

*and then, we have a decomposition into the three above 5-irreducible numerical semigroups of  $S$ . However, if we apply Algorithm 5:*

- $P(11, 22, 8, 14) = \{17\}$ ,
- $P(11, 22, 13, 9) = \{17\}$ ,
- $P(11, 17, 28, 14) = \{23\}$ ,

*while  $\{x \in \text{SG}(S) : x > 5\} = \{17, 23\}$ , so  $(11, 22, 13, 9)$  and  $(11, 17, 28, 14)$  are enough to decompose  $S$  into 5-irreducible numerical semigroups.*

In the above example, both 5-irreducible numerical semigroups in the decomposition are also irreducible. In the next example, we show that it is not true in general.

**Example 27.** *Let  $S = \{0, 9, 17, 18, 24, \rightarrow\}$ . The coordinates for  $S$  are  $x = (28, 29, 30, 31, 32, 24, 25, 17)$ . By using Algorithm 4, we obtain that the coordinates of the elements in  $\text{Minimals}_{\subseteq}(\mathcal{J}_m(S))$  are:*

$$\begin{aligned} &(10, 20, 21, 31, 14, 15, 16, 17), (10, 20, 12, 22, 32, 15, 16, 17), (19, 11, 21, 13, 32, 15, 16, 17), \\ &(19, 11, 30, 13, 14, 15, 16, 17), (19, 20, 12, 13, 32, 15, 16, 17), (28, 11, 12, 13, 14, 15, 16, 17), \\ &(10, 20, 30, 13, 14, 15, 16, 17), (10, 11, 12, 13, 14, 24, 16, 17), (19, 29, 12, 13, 14, 15, 16, 17), \\ &(10, 11, 12, 13, 14, 15, 25, 17), (10, 11, 21, 22, 32, 15, 16, 17) \text{ and } (19, 20, 12, 31, 14, 15, 16, 17), \end{aligned}$$

*and then, we have a decomposition into the above 9-irreducible numerical semigroups of  $S$ . However, if we apply Algorithm 5 we obtain these 12 minimal 9-irreducible oversemigroups of  $S$ :*

- $P(10, 20, 21, 31, 14, 15, 16, 17) = \{22\}$ ,
- $P(10, 20, 12, 22, 32, 15, 16, 17) = \{23\}$ ,
- $P(19, 11, 21, 13, 32, 15, 16, 17) = \{23\}$ ,
- $P(19, 11, 30, 13, 14, 15, 16, 17) = \{21\}$ ,



- $P(19, 20, 12, 13, 32, 15, 16, 17) = \{23\}$ ,
- $P(28, 11, 12, 13, 14, 15, 16, 17) = \{19\}$ ,
- $P(10, 20, 30, 13, 14, 15, 16, 17) = \{21\}$ ,
- $P(10, 11, 12, 13, 14, 24, 16, 17) = \{15\}$ ,
- $P(19, 29, 12, 13, 14, 15, 16, 17) = \{20\}$ ,
- $P(10, 11, 12, 13, 14, 15, 25, 17) = \{16\}$ ,
- $P(10, 11, 21, 22, 32, 15, 16, 17) = \{23\}$ ,
- $P(19, 20, 12, 31, 14, 15, 16, 17) = \{22\}$ .

Because  $\{x \in \text{SG}(S) : x > 9\} = \{15, 16, 19, 20, 21, 22, 23\}$ , then  $(10, 11, 12, 13, 14, 24, 16, 17)$ ,  $(10, 11, 12, 13, 14, 15, 25, 17)$ ,  $(28, 11, 12, 13, 14, 15, 16, 17)$ ,  $(19, 29, 12, 13, 14, 15, 16, 17)$ ,  $(19, 11, 30, 13, 14, 15, 16, 17)$ ,  $(10, 20, 21, 31, 14, 15, 16, 17)$  and  $(19, 11, 21, 13, 32, 15, 16, 17)$  are enough to decompose  $S$  into 9-irreducible numerical semigroups.

Note that the standard decomposition into irreducible numerical semigroups is given by the coordinates  $(9, 18, 19, 12, 13, 22, 31)$ ,  $(10, 11, 12, 13, 14, 15, 25, 17)$ ,  $(28, 11, 12, 13, 14, 15, 16, 17)$ ,  $(19, 11, 30, 13, 14, 15, 16, 17)$ ,  $(19, 29, 12, 13, 14, 15, 16, 17)$  and  $(19, 20, 12, 31, 14, 15, 16, 17)$ , where  $(9, 18, 19, 12, 13, 22, 31)$  has multiplicity 8, and then, it is not a decomposition into 9-irreducibles.

**Remark 28.** Note that analogously to the extension done in this paper for irreducible numerical semigroups when the multiplicity is fixed, we could also extend the notions of symmetry and pseudosymmetry as follows:

- $S \in \mathcal{S}(m)$  is  $m$ -symmetric if  $S$  is  $m$ -irreducible and  $F(S)$  is odd.
- $S \in \mathcal{S}(m)$  is  $m$ -pseudosymmetric if  $S$  is  $m$ -irreducible and  $F(S)$  is even.

Moreover, by using Proposition 6, we can describe, for a given multiplicity  $m$ , all the  $m$ -symmetric and  $m$ -pseudosymmetric numerical semigroups in terms of the Frobenius number. Denote by  $\text{Symm}(m)$  and  $\text{PSymm}(m)$  the set of  $m$ -symmetric and  $m$ -pseudosymmetric numerical semigroups, respectively. Let  $S \in \mathcal{S}(m)$  and  $F(S) = F$ .

- (1) If  $F = m - 1$ , then  $S \in \text{Symm}(m)$  (resp.  $S \in \text{PSymm}(m)$ ) if and only if  $S = \{x \in \mathbb{N} : x \geq m\} \cup \{0\}$  and  $m$  is even (resp.  $m$  is odd).
- (2) If  $m < F < 2m$ , then  $S \in \text{Symm}(m)$  (resp.  $S \in \text{PSymm}(m)$ ) if and only if  $S = \{x \in \mathbb{N} : x \geq m, x \neq F\} \cup \{0\}$  and  $F$  is odd (resp.  $F$  is even).
- (3) If  $F > 2m$ , then  $S \in \text{Symm}(m)$  (resp.  $S \in \text{PSymm}(m)$ ) if and only if  $S$  is symmetric (resp.  $S$  is pseudosymmetric).

## REFERENCES

- [1] Apéry, R. (1946). Sur les branches superlinéaires des courbes algébriques. C. R. Acad. Sci. Paris 222, 1198–2000.
- [2] Barucci, V., Dobbs, D.E., and Fontana, M. (1997). Maximality properties in numerical semigroups and applications to one-dimensional analitically irreducible local domains. Memoirs of the American Mathematical Society. Vol.125, n.598.
- [3] Barucci, V. and Froberg, R. (1997). One-dimensional almost Gorenstein rings. Journal of Algebra 188, p.418–442.
- [4] Kunz, E. (1973). The value-semigroup of a one-dimensional Gorenstein ring, Proc. Amer. Math. Soc. 25 (1973), 748–751.
- [5] Rosales, J.C. and Branco, M.B. (2002). Decomposition of a numerical semigroup as an intersection of irreducible numerical semigroups. B. Belg. Math. Soc-Sim. 9 (2002), 373–381.
- [6] Rosales, J.C. and Branco, M.B. (2002). Numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups. J. Pure Appl. Algebra 171 (2-3) (2002), 303–314.
- [7] Rosales, J.C., García-Sánchez, P.A., García-García, J.I. and Jimenez-Madrid, J.A. (2003). The oversemigroups of a numerical semigroup. Semigroup Forum 67 (2003), 145–158.
- [8] Rosales, J.C. and Branco, M.B. (2003). Irreducible numerical semigroups, Pacific J. Math. 209 (2003), 131–143.
- [9] Rosales, J.C. and García-Sánchez, P.A. (2009). Numerical semigroups, Springer, New York, NY, 2009. ISBN: 978-1-4419-0159-0.

DEPARTAMENTO DE ÁLGEBRA. UNIVERSIDAD DE GRANADA  
E-mail address: <sup>†</sup>vblanco@ugr.es; <sup>‡</sup>jrosales@ugr.es